An Introduction to The Theory of Equations

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Introduction

The Theory of Equations is a diverse area which encompasses many topics in mathematics, including complex numbers, constructions, cubic and quartic equations, graphs and symmetric functions. The Theory of Equations is sometimes studied from a textbook in a classroom on the college level, however in recent years the topics included in this field have been addressed in high school and college algebra classes. Therefore, this area has lost some of the exposure that it once had. Nevertheless, the topics dealt with in the theory of equations are important and exciting. I decided personally to focus on this sphere of mathematics because it is interesting and relevant to teaching at the high school level. The subjects will be proved for the benefit of teachers, and methods will be given to simplify the material for students.

The topics included in this paper are cubic equations, synthetic division, the remainder theorem and Descartes' rule of signs. Synthetic division and the remainder theorem are topics which are usually covered in high school. Even though I have dealt with cubic equations in graphing, my experience with them has been somewhat limited. Descartes' rule of signs is a concept which I find intriguing and insightful; it gives the superior limit for the number of positive and negative roots of an equation. Each of the topics will be proved and explored. Then each will be discussed in one context in which they can be used in a high school classroom; what should be taught, who the material should be given to and lesson plans will be presented.

This paper will also try to answer various mathematical questions such as “Can a cubic equation be solved by radicals?”, and “Isn’t there an easier way to divide polynomials?”. Proving these theorems and exploring them has given the author a great deal of respect for the mathematicians who proved them originally and for the people who type in the subscripts and powers when writing equations. We will begin our study with Synthetic Division, an easier topic which seems to thread it’s way through the rest of the topics.

Synthetic Division

Ordinary long division of a polynomial by a binomial in the form x-c can be shortened by using synthetic division. An example of long division is

$$\begin{array}{r l}
 & 2x^2 + x + 5 \quad \text{quotient} \\
x-2 & \overline{\big) 2x^3 - 3x^2 + 3x - 4} \\
 & \underline{2x^3 - 4x^2} \\
 & x^2 + 3x \\
 & \underline{x^2 - 2x} \\
 & 5x - 4 \\
 & \underline{5x - 10} \\
 & 6 \quad \text{remainder is 6}
\end{array}$$

Even though this is not a tremendous amount of work, sometimes it can be difficult and long. Synthetic division provides a more convenient technique and is easier to use. To prove that the technique yields the same quotient as long division let's look at the identity.

$$f(x) = (x - c)q(x) + r$$

Now substitute $q(x) = b_0x^{n-1} + b_1x^{n-2} + \ldots\ldots + b_{n-1}$ where b_0, b_1, ... b_{n-1} are coefficients to be determined. Multiplying we have

$$(x-c)q(x) = b_0x^n + b_1x^{n-1} + \ldots + b_{n-1}x - cb_0x^{n-1} - cb_1x^{n-2} - \ldots - cb_{n-1}$$
$$= b_0x^n + (b_1 - cb_0)x^{n-1} + (b_2 - cb_1)x^{n-2} + \ldots + (b_{n-1} - cb_{n-2}) - cb_{n-1}$$

and

$$(x-c)\ q(x) + r = b_0x^n + (b_1 - cb_0)x^{n-1} + \ldots + (b_{n-1} - cb_{n-2}) + r - cb_{n-1}$$

Now this polynomial must be identical to some

$a_0x^n + a_1x^{n-1} + \ldots\ldots + a_n$ which is f(x) .

To determine $b_0, b_1, \ldots, b_{n-1}$ and r we set coefficients of like powers of x equal to each other. Now we have the set of equations $b_0 = a_0$, $b_1 - cb_0 = a_1$, $b_2 - cb_1 = a_2$, $b_{n-1} - cb_{n-2} = a_{n-1}$, $r - cb_{n-1} = a_n$. Now we need only solve for the b's and r. So $b_{n-1} = a_{n-1} + cb_{n-2}$ and $r = a_n + cb_{n-1}$. The reader might notice that the calculations keep repeating. Synthetic Division can now be written in a more convenient way:

c] a_0	a_1	$a_2 \ldots$	a_{n-1}	a_n	
	b_0c	$b_1c\ldots$	$b_{n-2}c$	$b_{n-1}c$	
$a_0 = b_0$	b_1	$b_2 \ldots$	b_{n-1} r		remainder

In the first line all the coefficients of f(x) are written without omission starting with a_0. The third line begins with $b_0 = a_0$, which is then multiplied by c, the product is then placed in the second line and added to a_1; the sum b_1 is placed in the third line. Again, b_1 is multiplied by c, the product is placed in the second line and added to a_2; the sum is place in the third line and the same process is repeated until in the last column and third line the remainder is found. The expressions for $b_0, b_1, \ldots, b_{n-1}$, r obtained by substitutions are:

$b_0 = a_0$, $b_1 = a_0c + a_1$, $b_2 = a_0c^2 + a_1c + a_2$, $b_{n-1} = a_0c^{n-1} + a_1c^{n-2} + \ldots +$

a_{n-1} and $r = a_0c^n + a_1c^{n-1} + ...+a_n = f(c)$

Since the remainder is f(c), synthetic division provides a way for calculating the value of a polynomial for a given value of the variable. Now we will go back to the original example we did with long division and use synthetic division.

Example: Divide $2x^3 - 3x^2 + 3x - 4$ by $x-2$

2]	2	-3	3	-4
		4	2	10
	2	1	5	6

So the quotient is $2x^2 + x + 5$ with a remainder of 6.

The first power of the quotient is always equal to one less than the highest power of the polynomial we are dividing, this is true because if the dividend is of the nth degree and the divisor of the 1st degree then the quotient is of the n-1 degree. Now let us consider another more difficult problem.

Example: Find quotient and the remainder when dividing
$3x^6 - 7x^5 + 5x^4 - x^2 -6x-8$ by $x+2$

-2]	3	-7	5	0	-1	-6	-8
		-6	26	-62	124	-246	504
	3	-13	31	-62	123	-252	496

Therefore, the quotient is

$$3x^5 - 13x^4 + 31x^3 - 62x^2 + 123x - 252$$

and the remainder is 496.

As illustrated in the example if there is a missing power of the variable in the polynomial then simply put a zero for that coefficient in the first line. Likewise, if a zero appears in the third line then the term where the zero appears has a coefficient of zero and may be omitted when writing the quotient.

When teaching synthetic division in the classroom students should be told a few things: synthetic division is an easier process than long division, it performs the same operation of long division and learning synthetic division will be helpful in the future. When presenting synthetic division a problem should be done in long division and then compared to one in synthetic division. From here, the teacher should proceed along the same lines as illustrated in this paper, omitting the proof. It should be kept in mind that students should be able to multiply, add, and subtract integers before doing synthetic division. This topic will be used throughout an algebra course so students will constantly be reviewing synthetic division.

Remainder Theorem

Synthetic division and the remainder theorem are closely linked because they both involve division of a polynomial by a binomial $x - c$. The remainder in the division by $x - c$, where c is an arbitrary number can be found without actually performing the division.

The Remainder Theorem. The remainder obtained in dividing f(x) by x-c is the value of the polynomial f(x) for x=c, that is f(c).

To prove this theorem once again we use the identity

$$f(x) = (x - c)q(x) + r$$

When we substitute the number c in place of x into the identity we must arrive at equal numbers. Because r is constant it of course is not affected by this substitution.

$$f(c) = (c-c)q(r) + r$$

therefore

$$f(c) = r$$

which also means that in x

$$f(x) = (x-c)q(x) + f(c)$$

Using this theorem it follows that f(x) is divisible by a binomial x-c if and only if f(c) =0

$$f(x) = (x-c)q(x) + 0$$

$$f(x) = (x-c)q(x)$$

which follows that $x - c$ is a factor of f(x).

Now we will do a few examples.

Example: Show that $f(x) = x^3 + x^2 - 5x + 3$ is divisible by $x + 3$.

In this case $c = -3$, and thus we calculate

$$f(-3) = -27 + 9 + 15 + 3 = 0$$

therefore $f(x)$ is divisible by $x + 3$.

Example: Show that $x^n - c^n$ is divisible by $x - c$.

We know this is true since $c^n - c^n = 0$

Example: Under what conditions is $x^n + c^n$ divisible by $x + c$?

The result of the substitution $x = -c$ is

$$(-c)^n + c^n = c^n + c^n = 2c^n \text{ if } n \text{ is even}$$

$$(-c)^n + c^n = -c^n + c^n = 0 \text{ if } n \text{ is odd}$$

Hence, $x^n + c^n$ is divisible by $x+c$ (for $c \neq 0$) only if n is odd, and for an even n the remainder after the division is $2c^n$.

Applying the factor theorem, the remainder theorem and synthetic division,students, and more importantly, teachers can write equations with specified roots.

For example:

Write an equation having only 5,1,-3 as roots.

Using the factor theorem we know that (x-5)(x-1)(x+3) are factors of the equation and since there are only three roots the equation is of the 3rd degree. So,

$$(x-5)(x-1)(x+3) = 0 \text{ or } x^3 - 3x^2 - 13x + 15 = 0$$

Using the preceding theorems teachers could design problems in which they want students to have experience finding certain roots. For example if a teacher wanted to design a problem with 0 and 1 + 5i as roots so students could work with imaginary numbers all that he or she would have to do is multiply (x-0)(x -(1 +5i)

$$x[\, x - (1 + 5i)][\, x - (\, 1 - 5i)] = x[(\, x-1\,) - 5i]\, [(x-1) + 5i] = x[(x-1)^2 + 25]$$

$$= x(x^2 - 2x + 26) = 0 \text{ or } x^3 - 2x^2 + 26x = 0$$

The topics discussed so far can thus aid the teacher in designing test and quiz questions for problems in which they already know the roots.

If the remainder theorem is not addressed in the students textbook it is a subject to bring into an algebra class. When using the remainder theorem in the classroom there are a few things with which the students should already be familiar. The students should be proficient in algebra, and in particular, in factoring polynomials. The students should also have a grasp on division of polynomials. The remainder theorem and factor theorem will be used to teach students more about polynomials and their roots. The teacher should introduce this topic when students are learning to do long division. When explaining the remainder theorem a short proof could be done, for example, showing that by definition

$$f(x) = (x-c)q(x) + R.$$

Where $f(x)$ is the dividend, $(x-c)$ is the divisor, $q(x)$ is the quotient and R is equal to the remainder. The parallel with division by integers should not be overlooked. To show this, polynomial division could be done along with number division. Now the the Socratic method of questioning could be useful here; the teacher can ask the students what might happen when $R = 0$. The students could be directed to a conclusion that if $R = 0$ then $(x-c)$ is a factor of $f(x)$, which is the factor theorem. Problems can be done on the board and students will be asked to participate in answering questions on the board. It will be explained that the remainder theorem can be used as a means to check work and help find out if a certain polynomial is divisible by a binomial. By explaining the remainder theorem and going through the proof the students will gain a deeper understanding of algebra. This topic should be addressed in class period and returned to throughout the semester. In ending this lesson, homework will be given and hopefully students will have time to work in class on problems so the teacher can answer any questions. This topic can be evaluated by testing on the theorem.

Descartes' Rule of Signs

In a sequence of numbers

$$a_0, a_1, a_2, \ldots, a_n$$

none of which is zero, two consecutive terms

$$a_{i-1} \text{ and } a_i$$

may have the same sign or opposite signs. If the terms a_{i-1}, a_i have the same sign then there is a permanence of signs and if they have opposite signs then they present a variation of signs. For example, in the sequence

$$-2, \quad -5, \quad 1, \quad 4, \quad -4, \quad 8, \quad 1, \quad 2, \quad -5, \quad -3, \quad 2$$

there are five variations and five permanences. If some terms in a sequence are 0, they are disregarded in counting the number of variations and permances. Therefore in the sequence

$$4, \quad 0, \quad 0, \quad 2, \quad 0, \quad -3, \quad -4, \quad 0, \quad 6, \quad 1, \quad 0, \quad 0, \quad -1$$

there are 3 permances and 3 variations. Using this terminology we can state

Descartes' Rule of Signs. The number of positive real roots of an equation with real coefficients

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots + a_n = 0$$

is never greater than the number of variations in the sequence of it's coefficients

$$a_0, a_1, \dots, a_0$$

and, if less, then always by an even number.

Now let's prove that this theorem is true by using mathematics induction. First let V denote the number of variations and r the number of positive roots, with each root counted according to the multiplicity. We want to prove that

$$V = r + 2h$$

where h is a nonnegative integer. If $V = 0$, then all the coefficients that are different from 0 are the same sign. Therefore, the equation has no positive roots so that $r = 0$.

Now let's assume that the theorem is true for $V - 1$ variations, then we shall prove that it is true in the case of V variations. First, let a_∂ and a_β, $\beta > \partial$, be two coefficients of opposite sign, with coefficients between being 0. The total number of variations, V,is composed of three parts:

the number of the variations v_1 in the section

$$a_0, a_1, \dots, a_\partial$$

one variation as assumed in the section

$$a_\partial, \dots, a_\beta$$

and the number of variations v_2 in the section

$$a_\partial, \dots, a_n$$

so that

$$V = v_1 + v_2 + 1 \text{ or } V-1 = v_1 + v_2$$

Now looking at a different equation

$$F(x) = xf'(x) - \Omega f(x) = 0$$

where $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$

$$f'(x) = na_0x^{n-1} + a_1(n-1)x^{n-2} + a_2(n-2)x^{n-3} + a_{n-1} = 0$$

$$xf'(x) = na_0\,x^n + (n-1)a_1x^{n-1} + (n-2)a_2x^{n-2} + \dots + a_{n-1}x = 0$$

$$\Omega f(x) = \Omega a_0x^n + \Omega a_1x^{n-1} + \Omega a_2x^{n-2} + \dots + \Omega a_{n-1}x + \Omega a_n = 0$$

Remember that $F(x) = xf'(x) - \Omega f(x)$ so

$$(n-\Omega)a_0x^n + (n-1-\Omega)a_1x^{n-1} + (n-2-\Omega)a_2x^{n-2} + (n-\partial-\Omega)a_\partial x^{n-\partial} + \dots +$$

$$(n-\beta-\Omega)a_\beta{}^{n-\beta} + \dots + -(\Omega)a_n = 0$$

now we choose Ω so that

$$n-\partial-\Omega > 0, \quad n-\beta-\Omega < 0$$

or $n - \beta < \Omega < n - \partial$. This is possible because it is true that $\beta > \partial$.

Now notice the factors $n-\Omega$, $n-1-\Omega$, ... , $n-\partial-\Omega$ are positive while the factors $n-\beta-\Omega$, $n-\beta-1-\Omega$, $n-n-\Omega = -\Omega$ are negative. In the sections

$$(n-\Omega)a_0, \dots, (n-\partial-\Omega)a_\partial$$

and $$(n-\beta-\Omega)a_\beta, \dots, (-\Omega)a_n$$

we count v_1 and v_2 variations, but in the sections

$$(n-\partial-\Omega)a_\partial, \ldots, (n-\beta-\Omega)a_\beta$$

there is no variation since the extreme terms are of the same sign and intermediate terms are zero. Thus for the equation

$$F(x) = xf'(x) - \Omega f(x) = 0$$

the number of variations is $\nu_1 + \nu_2 = V - 1$.

We use de Gua's theorem which says that the equation

$$F(x) = xf'(x) + \partial f(x) = 0$$

has at least r - 1 positive roots if the polynomial f(x) has r postive roots. By de Gua's theorem and assuming once again that the theorem is true in the case of V-1 variations, we have

$$r - 1 \leq V - 1$$

where $$r \leq V.$$

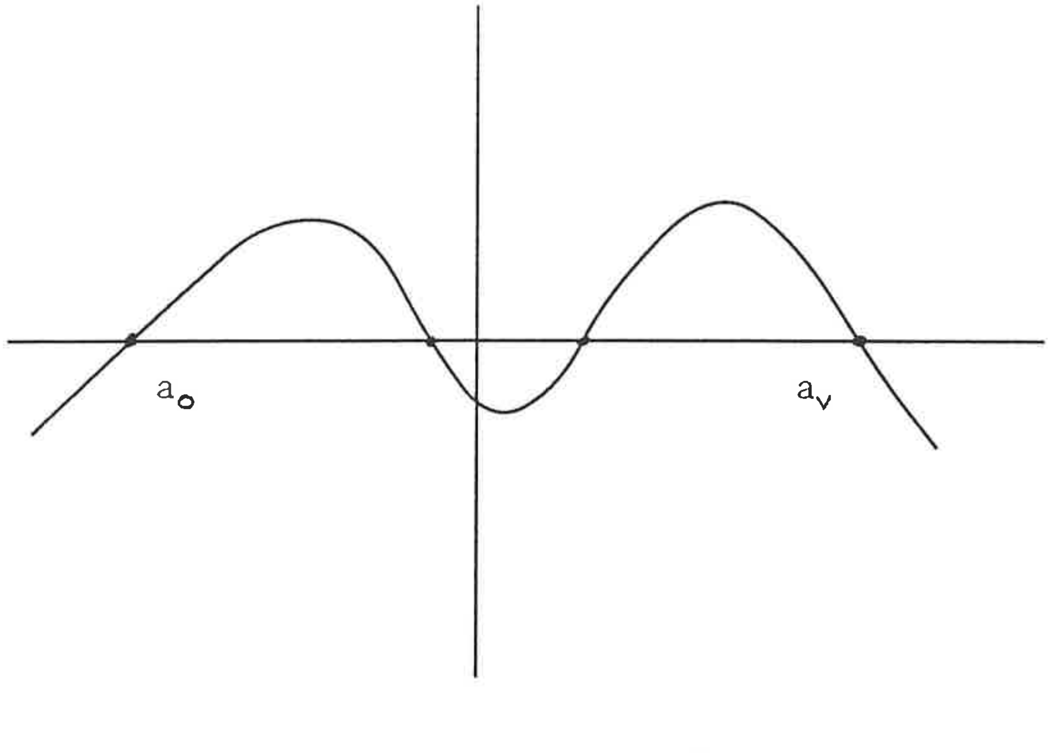
Now we must show that the number of roots, if not equal to the variations is then less by an even number. Thus, we must prove that the difference $V - r$ is an even number. In the sequence

$$a_0, a_1, \ldots, a_n$$

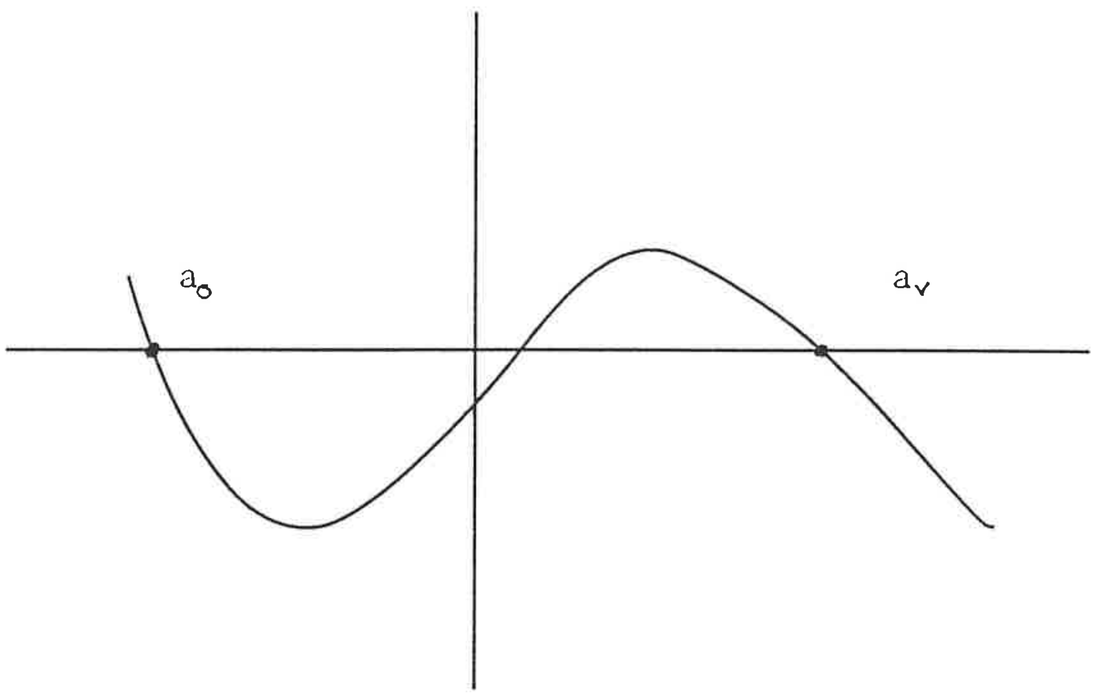
let a_ν be the last term that is not zero. If V is even, then a_ν and a_0 have like signs, and opposite signs if V is odd. For example $-a_0, a_1, -a_2, a_\nu$ has three variations while $-a_0, a_1, -a_\nu$ has two variations. The polynomial

$$f(x) = a_0x^n + \ldots\ldots\ldots + a_\nu x^{n-\nu}$$

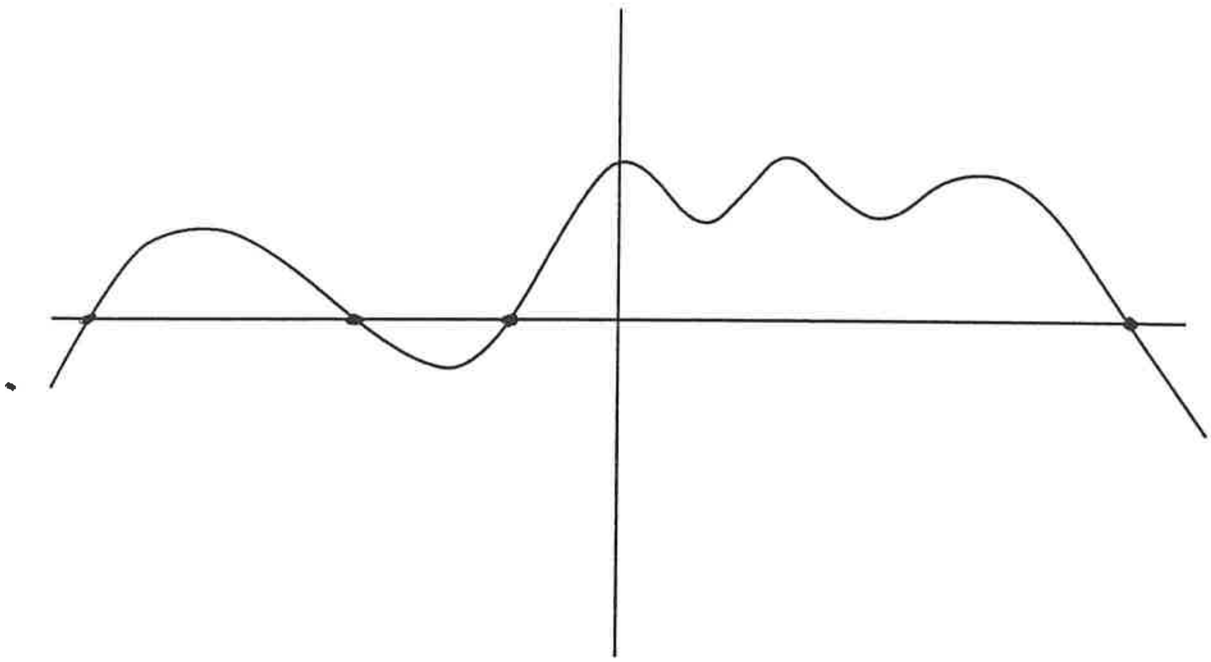
has the sign of a_ν for a small positive x, and for a large positive x the sign of a_0. This is easy to understand for if x is small the a_0 term would decrease more significantly than the a_ν term. Thus if a_0 and a_ν have the same sign, the number of positive roots is even, the same as ν, and if a_0 and a_ν have opposite signs, the number of such roots is odd , the same as ν. By constructing a few graphs it will be easier to show this.



In this case a and a have the same sign. The points are marked where there are roots. Therefore, there is an even number of roots, as illustrated. There are also four variations.



In this case a and a have different signs. There is an odd number of roots (three) and three variations.



This graph illustrates a very fundamental part of Descartes' rule which says that if the number of roots is not equal to the number of variations then it is less thatn by an even number. The reason that the number of roots is less thatn the number of variation is that if the line does not cross the x-axis then it is not a root however a variation can occur and if the line does not cross the x-axis then it must have two variations for every time it does not cross the x-axis.

Therefore, r and V are together even or odd, and their difference V-r is an even number.

There is also a way that we can find the number of negative real roots. By changing x into -x in the equation f(x) = 0, we get another equation f(-x) = 0, which has as many positive roots as the proposed equation has negative roots. So, r' is the number of negative roots of the proposed equation, and V' the number of variations in f(-x), then

$$V' = r' + 2h'$$

where h' is a nonnegative integer.

Descartes' Rule of Signs gives the exact number of positive real roots in two cases: V = 0 and V = 1. In the first case r = 0

$$r + 2h = 0$$

$$r = 0 \quad \text{since } h \text{ is nonnegative}$$

and in the second relation $r + 2h = 1$

because h is a nonnegative integer this requires h = 0 and r = 1. This particular result can be proved as follows: If there is only one variation in the sequence

$$a_0, a_1, \ldots, a_n$$

it can be divided into two parts: the first $a_0, a_1, \ldots, a_{u-1}$ consisting of positive or zero terms and the second $a_u = -b_0$, $a_{u+1} = -b_1$, ..., $a_n = -b_{n-u}$ starting with a negative term a_u and consisting of negative and zero terms. The polynomial f(x) can be presented as such:

$$f(x) = x^{n-u+1}[a_0x^{u-1} + ... + a_{u-1} -(b_0/x + ... + b_{n-u}/x^{n-u+1})].$$

where the expression in the brackets is the difference between an increasing and decreasing function. It so happens that f(x) itself is an increasing function. Now if we were to substitute a very large negative value (which would make x small and positive), this would pass to a very large positive value (where x is large and positive). Therefore it goes through 0 only once, giving only one positive root of the equation f(x) = 0.

Sometimes V > 1, in this case Descartes' rule indicates only and upper limit to the number of positive and negative roots and sometimes reveals the presence of imaginary roots, as we shall soon see.

Example: How many real roots does $f(x) = x^4 + x^2 - x - 3 = 0$ have?

f(x) presents only one variation so there is one positive root. The coefficients of f(-x) have the following signs +,+,+,- so there is only one variation, therefore one negative root. This tells us that there are two imaginary roots.

Example: How many real roots in $f(x) = x^6 - x^3 + 2x^2 - 3x - 1 = 0$?

For this equation V = 3, so there may be one or three positive roots. Changing x into -x the coefficients become +,+,+,+,- so

$V = 1$, and there is just one negative root of the proposed equation. The total number of real roots is not greater than four, and so at least two roots are imaginary.

Because imaginary roots occur in pairs it makes it a bit easier to use Descartes' rule of signs. If an equation has a root where $r_1 = a+bi$ then it also has a root $r_2 = a - bi$. Therefore, if we have an equation of the 6th degree with at most 4 positive roots and 1 negative root then we know that there are at least two imaginary roots because the equation has only a total of five real roots.

Using Descartes' Rule of signs and Synthetic Division one can find the rational roots of an equation. If a rational number b/c is in its lowest terms and is a root of an equation then b is a factor of a_n and c is a factor of a_0 in the equation of the form

$$a_0x^n + a_1x^{n-1} + \dots + a_n = 0$$

To solve the equation we first try to find the maximum number of positive roots and negative roots. Next, we try to find the rational roots by depressing the equation and factoring out the roots. We can continue this process until arriving at all the roots or merely arrive at a quadratic equation and then solve by the quadratic formula.

Example: Solve this equation $8x^4 - 18x^3 + 2x^2 + 7x - 6 = 0$

$f(x)$ = + - + + - = 3 variatons $f(-x)$ = + + + -- = 1 variation

So now we know that at most there are four real roots and with three positive and one negative. Now we find the factors of a_n and a_0.

a_n = -6,1,-1,6,2,-3,-2,3 a_0 = -1,1,-8,8,-2,2,4,-4

Therefore b/c = 6,-6,3/4,-3/4,3,-3,3/2,-3/2,1,-1,1/8,-1/8,2,-2,1/2, -1/2,1/4,-1/4,3/8,-3/8. First we will try integers and try to find f(x) = 0 by synthetic division.

2]	8	-18	2	7	-6
		16	-4	-4	6
	8	-2	-2	3	0

therefore (x-2) is a factor of f(x). Next we try to divide by another number, this time use the newly found cubic equation we have.

-3/4]	8	-2	-2	3
		-6	6	-3
	8	-8	4	0

Now we know that (x+3/4) is also a factor of the original quartic equation. From here we simply have to substitute in the coefficients of our new quadratic equation into the quadratic formula and we get

$$\frac{2 \pm \sqrt{-4}}{4} = \frac{1 \pm i}{2}$$

Therefore the roots of the equation $8x^4 - 18x^3 + 2x^2 + 7x - 6 = 0$ are

2,-3/4, $\frac{1 \pm i}{2}$.

Using the rational roots theorem now we can find the roots of polynomials with very high degrees. Even though using rational roots can be a bit lengthy, it does work. Using this procedure we can check our cubic roots that we will now learn how to find.

Cubic Equations

One of the common problems in algebra is finding the solution of algebraic equations. The solution of an equation is the determination of all its roots, either real or imaginary. The difficulty of solving equations naturally increases with their degree because the higher the degree the more roots there are to compute. For equations of the first degree

$$ax + b = 0$$

the solution for x is given by the formula

$$x = -b/a$$

which shows that arithmetic operations must be performed on arbitrary coefficients. The solution of equations of the second degree, $ax^2 + bx + c = 0$, can be found using the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which once again shows that the solution of a quadratic equation can be determined by coefficients.

What about polynomial equations of degree 3 ? Can a general solution be found just using the coefficients of an equation in the form of

$$ax^3 + bx^2 + cx + d = 0 ?$$

A general formula for the solution refers to some definite procedure for calculating the solutions from the coefficients of the polynomials. To do this one must use rational operations including addition, subtraction, multiplication and division and the extraction of roots. When using these operations the solutions are called solutions by radicals. So, can equations of degree 3, and even equations with the degree higher than three be solved by performing rational operations on their coefficients?

People's interest in cubic equations can be traced back to the times of the early Babylonians, about 1800 - 1600 B.C. However, the algebraic solution of the third degree and the fourth degree equations is a product of the Italian Renaissance. It was shown by Italian Algebraists, Scipione Ferro and Ludovico Ferrari, that the cubic and quartic equations can be solved algebraically and their roots presented in radical form for random arbitrary values of the coefficients. It so happens that for polynomials of degree less than or equal to four a procedure exists; however, for equations of higher degree, no such procedure exists.

In this paper a solution of roots using the solution by radicals for equations of the third degree will be addressed. Initially an attempt was made to come up with a formula in which the student could substitute in the coefficients of a 3rd degree equation (much the same way as the quadratic formula works) and come up with three roots. It appears this should be a simple enough task. All that is nececessary is a little algebra and the patience to keep working on the problem. Wrong! After much deliberation and many pages of work I arrived at a formula that was extremely long and incorrect. Because of the fact that I could find no general formula using coefficients in any texts and could not come up with one myself, I chose to use Cardan's Formula to give the roots of an equation with the degree of three.

First I want to show that a cubic equation indeed can be solved by radicals. Taking the general cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

and without loss of generality, let $a = 1$. If $a \neq 1$ then dividing by a would guarantee the leading coefficient to be equal to one. Since the equation is a cubic, $a \neq 0$ because if that were the case we would have a quadratic equation which is solvable by radicals. Next substiting $x = y + L$ into the general cubic equation it becomes

$(y + L)^3 + b(y + L)^2 + c(y + L) + d = 0$

$y3 + 3y^2L + 3yL^2 + L^3 + b(y^2 + 2yL + L^2) + c(y + L) + d = 0$

$y^3 + y^2(3L + b) + y(3L^2 + 2bL + c) + (L^3 + bL^2 + cL + d) = 0$

Now let $L = -b/3$ so that the y^2 term drops out.

$y^3 + y^2(-3b/3 + b) + y(3b^2/9 - 2b^2/3 + c) + (-b^3/27 + b^3/9 - cb/3 + d) = 0$

Now there is an equation in the form:

$$y^3 + By + C = 0$$

where :

$B = b^2/3 - 2b^2/3 + c = -b^2/3 + c$

$C = 2b^3/27 - cb/3 + d$

If $C = 0$ then the solution becomes $y(y^2 + B) = 0$

$$y = 0, \qquad y^2 = -B = b^2/3 - c$$

$$y = \pm\sqrt{\frac{b^2 - 3c}{3}}$$

$$y_1 = 0 \qquad y_2 = \sqrt{-B} \qquad y_3 = -\sqrt{-B}$$

Since $x = y + L$ with $L = -b/3$ then $x - L = y$, which gives us $x + b/3 = y$.

To obtain x we have to substitute values,

$$x_1 = -b/3 \qquad x_2 = -b/3 + \sqrt{\frac{b^2 - 3c}{3}} \qquad x_3 = -b/3 - \sqrt{\frac{b^2 - 3c}{3}}$$

If $C \neq 0$, then $y = 0$ is not a root of the equation. If that were the case then $y^3 + By + C = 0$

$0 + 0 + C = 0$, giving us $C = 0$ which contradicts the assumption. It still needs to be proven that this equation can be solved by radicals so we let $y = Z + K/Z$ and substitute this for y into the equation $y^3 + By + C = 0$ which leaves the following equation:

$(Z + K/Z)^3 + B(Z + K/Z) + C = 0$

$Z^6 + Z^4(3K + B) + CZ^3 + Z^2(3K^2 + BK) + K^3 = 0$

By letting $K = -B/3$ the equation can be simplified

$Z^6 + Z^4(-3B/3 + B) + CZ^3 + Z^2(3B^2/9 - B^2/3) + K^3 = 0$

Two terms drop out leaving the equation

$Z^6 + CZ^3 + K^3 = 0$ OR $(Z^3)^2 + C(Z^3) + K^3 = 0$

By letting $Q=Z^3$ we then have the quadratic equation $Q^2 + CQ + K^3 = 0$. This equation can be solved by the quadratic formula where

$$Z^3 = \frac{- C \pm \sqrt{C^2 - 3K^3}}{2} \qquad \text{or } Z = \sqrt[3]{\frac{-C \pm \sqrt{C^2 - 4K^3}}{2}}.$$

Now that there is a way to solve for Z all one must do is go back in the substitution to find x. Therefore it has been shown that a cubic equation can be solved by radicals. This is one point where the author tried substituting and working back to find some formula where we could just substitute. However, after being constantly stumped and producing an equation which was quite lengthy, a solution could not be found by just substituting for the coefficients. So, from here Cardan's Formula for an algebraic solution of cubic equations was investigated. At first we will go along much the same lines as we did to prove that a cubic could be solved by radicals; however, the process will be carried out longer until we can find the roots of the cubic.

Now, consider the general cubic equation once again:

$ax^3 + bx^2 + cx + d = 0$, where a,b,c,d are arbitrary real numbers where $a \neq 0$.

This equation can be reduced to a simpler form without the second degree term by making the transformation $x = y - b/3a$. Therefore we have:

$$a(y - b/3a)^3 + b(y-b/3a)^2 + c(y-b/3a) + d = 0$$

$$a(y^3 - y^2b/a + yb^2/3a^2 - b^3/27a^3) + b(y^2 - 2by/3a + b^2/9a^2) + c(y-b/3a) + d = 0$$

$$ay^3 + (b^2/3a - 2b^2/3a + c)y + (-b^3/27a^2 + b^3/9a^2 - bc/3a + d) = 0$$

Now we let:

$$B = b^2/3a - 2b^2/3a + c$$

$$C = -b^3/27a^2 + b^3/9a^2 - bc/3a + d = 2b^3/27a^2 - bc/3a + d$$

The general equation will become $ay^3 + By + C = 0$. The reader might notice that B and C are identical to the B and C used to prove that a cubic equation can be solved by radicals with the exception that $a = 1$ in the prior proof. To avoid fractions in our new general equation we will divide through by a and end up with

$$y^3 + 3py + 2q = 0$$

where $3p = B/a$ and $2q = C/a$.

This last equation is called the reduced cubic equation. As just illustrated here any cubic equation can be reduced to this form.

To solve this reduced equation we can look at the following identity:

$$(a + b)^3 - 3ab(\, a+b \,) - (a^3 + b^3) = 0$$

where the squared term is missing from the identity because we already eliminated the second term to produce the reduced equation.

Comparing the identity with the reduced equation we have that $a+b = y$, $ab = -p$ and $a^3 + b^3 = -2q$. From these equations we only have to find values for a and b to find y. This can be done by solving the following system:

$$a^3b^3 = -p^3$$

$$a^3 + b^3 = -\,2q$$

Now $b^3 = -2L - a^3$, and by substituting this value in the first equation, we have

$$-a^3(\, 2q + a^3) = -p^3$$

where $-a^3 2q - a^6 + p^3 = 0$ or $a^6 + a^3 2q - p^3 = 0$.

If we let $a^3 = v$, we obtain the following quadratic equation :

$$v^2 + 2qv - p^3 = 0$$

whose roots are:

$$v_1 = -q + \sqrt{q^2 + p^3}$$

$$v_2 = -q - \sqrt{q^2 + p^3}$$

Due to the symmetry of a and b in the system, we can take v_1 or v_2 to be a^3 or b^3 randomly. So, $a^3 = -q + \sqrt{q^2 + p^3}$ and $b^3 = -q - \sqrt{q^2 + p^3}$ where

$$a = \sqrt[3]{-q + \sqrt{q^2 + p^3}}$$

$$b = \sqrt[3]{-q - \sqrt{q^2 + p^3}}.$$

However, since $y = a + b$

$$y = \sqrt[3]{-q + \sqrt{q^2 + p^3}} + \sqrt[3]{-q - \sqrt{q^2 + p^3}}$$

which is called Cardan's formula for the cubic. Because a^3 and b^3 have three roots each, it seems that y has nine roots. However, this is not the case since $ab = -p$, the cubic roots of a^3 and b^3 are to be taken in pairs so their product (which is ab) is a rational number - p. Now the cubic roots of a^3 are : a (the principal value) , $a\partial$, and $a\partial^2$, where ∂ is one of the complex roots of unity. Similarily, the cubic roots of b^3 are: b, $b\partial$ and $b\partial^2$. The cubic root of unity is

$$\partial = \cos 2\pi/3 + i \sin 2\pi/3$$

which satisfies the equation

$$x^2 + x + 1 = 0.$$

Roots of this equation found algabraically are

$-1/2 + i\sqrt{3/2}$ and $-1/2 - i\sqrt{3/2}$.

However, it the product of a and b must be rational, the only admissible solutions are ab, $(a\partial)(b\partial^2) = ab\partial^3 = ab = -p$

(because $\partial^3 = -1$) , $(a\partial^2)(b\partial) = ab\partial^3$ or $-ab$

Therefore, the values of y are:

$$a+b,\ a\partial + b\partial^2, \text{ and } a\partial^2 + b\partial$$

But $x = y - b/3a$ and so the roots of the general cubic equation will be found once we know y. This knowledge should be sufficient to work out and find all roots. It will be addressed later on in the paper whether or not high school students will find complex roots.

Now consider the following example to illustrate Cardan's formula:

Example 1: Solve the equation $x^3 + 3x^2 + 9x - 13 = 0$

First we must reduce the equation to eliminate the second term. To transform let $x = y - b/3a$. Here, $a=1, b=3, c=9, d=13$.

$$x = y - 3/3 = y-1$$

Thus, substituting $y-1$ for x we have

$$(y-1)^3 + 3(y-1)^2 + 9(y-1) - 13 = 0$$

$$(y^3 - 3y^2 + 3y - 1) + 3(y^2 - 2y + 1) + 9(y-1) - 13 = 0$$

where the reduced equation becomes

$$y^3 + 6y - 20 = 0$$

Therefore $3p = 6,\ p = 2, \text{ and } p^3 = 8$

$2q = -20,\ q = -10, \text{ and } q^2 = 100$

Thus, $\sqrt{q^2 + p^3} = \sqrt{108} = 6\sqrt{3}$ and,

$$a = \sqrt[3]{10 + 6\sqrt{3}} = \sqrt[3]{1 + 3\sqrt{3} + 9 - 3\sqrt{3}} = \sqrt{(1 + \sqrt{3})^3} = 1 + \sqrt{3}$$

$$b = \sqrt[3]{10 - 6\sqrt{3}} = \sqrt[3]{1 - 3\sqrt{3} + 9 - 3\sqrt{3}} = \sqrt{(1 - \sqrt{3})^3} = 1 - \sqrt{3}$$

The solutions for the reduced equation are then,

$$y_1 = a + b = (1 + \sqrt{3}) + (1 - \sqrt{3}) = 2$$

$$y_2 = a\partial + b\partial^2 = (1 + \sqrt{3})(-1/2 + \sqrt{3}/2(i)) + (1 - \sqrt{3})(-1/2 - \sqrt{3}/2(i)) = -1 + 3i$$

$$y_3 = a\partial^2 + b\partial = (1 + \sqrt{3})(-1/2 - \sqrt{3}/2(i)) + (1 - \sqrt{3})(-1/2 + \sqrt{3}/2(i))$$

$$= -1 - 3i$$

But $x = y-1$. Therefore:

$$x_1 = y_1 - 1 = 2 - 1 = 1$$

$$x_2 = y_2 - 1 = -1 + 3i - 1 = -2 + 3i$$

$x_3 = y_3 - 1 = -1 - 3i - 1 = -2 - 3i$

Now, in this example we have one real solution and two conjugate complex roots.

There are different cases of solutions, reducible and irreducible. We now will look at these cases. Looking at Cardan's Formula it is apparent that the nature of the solutions will depend on the value of $q^2 + p^3$. Therefore it is called the discriminant of the cubic. This happens since $q^2 + p^3$ is under a square root sign ; it will yield real or imaginary values according to the sign of the sum q^2+p^3. If $q^2 + p^3 > 0$, then a and b will have one real value. From this , a + b and a - b will be real also. Therefore the solutions of the reduced equation are: $a+b = m$, $a-b = n$, where $y_1 = a + b = m$,

$y_2 = -m/2 + (n/2\sqrt{3})(i)$, $y_3 = -m/2 -(n/2\sqrt{3})(i)$.

Thus if $q^2 + p^3 > 0$, we have one real root and two imaginary roots.

Example: Solve $x^3 - 6x^2 + 10x - 8 = 0$

Eliminate squared term $y - (-6/3) = y + 2$. Where $x = y - b/3a$

Thus, substituting y + 2 for x in the equation:

$$(y+2)^3 - 6(y+2)^2 + 10(y+2) - 8 = 0$$

$$y^3 + 6y^2 + 12y + 8 - 6y^2 - 24y - 24 + 10y + 20 - 8 = 0$$

$$y^3 - 2y - 4 = 0 \quad \text{(Reduced equation)}$$

Therefore:

$3p = -2$, $p = -2/3$ and $p^3 = -8/27$

$2q = -4$, $q = -2$ and $q^2 = 4$

Thus, $q^2 + p^3 = 4 - 8/27 = 100/27 > 0$.

Hence, we know that in the solution one root must be real and two conjugate imaginary. The values for a and b are:

$$a = \sqrt[3]{-q + \sqrt{q^2 + p^3}} = \sqrt[3]{2 + \sqrt{100/27}} = \sqrt[3]{2 + 10/3\sqrt{3}}$$

$$b = \sqrt[3]{-q - \sqrt{q^2 + p^3}} = \sqrt[3]{2 - \sqrt{100/27}} = \sqrt[3]{2 - 10/3\sqrt{3}}$$

and simplifying:

$$a = \sqrt[3]{(6\sqrt{3} + 10)/3\sqrt{3}} = \sqrt[3]{(3\sqrt{3} + 9 + 3\sqrt{3} + 1)/\sqrt{27}} = \sqrt[3]{(\sqrt{3} + 1)^3/\sqrt{27}}$$

$$b = \sqrt[3]{(6\sqrt{3} - 10)/3\sqrt{3}} = \sqrt[3]{(3\sqrt{3} - 9 + 3\sqrt{3} - 1)/\sqrt{27}} = \sqrt[3]{(\sqrt{3} - 1)^3/\sqrt{27}}$$

$a = (\sqrt{3} + 1)/\sqrt{3}$ and $b = (\sqrt{3} - 1)/\sqrt{3}$

The solutions for the reduced equation are then:

$$y_1 = a + b = (\sqrt{3} + 1)/\sqrt{3} + (\sqrt{3} - 1)/\sqrt{3} = 2$$

$$y_2 = a\partial + b\partial^2 = ((\sqrt{3} + 1)/\sqrt{3})(-1/2 + \sqrt{3}/2\,(i)) + ((\sqrt{3} - 1)/\sqrt{3})(-1/2 - \sqrt{3}/2(i))$$

$$y_3 = a\partial^2 + b\partial = ((\sqrt{3} + 1)/\sqrt{3})(-1/2 - \sqrt{3}/2\,(i)) + ((\sqrt{3} - 1)/\sqrt{3})(-1/2 + \sqrt{3}/2\,(i))$$

and simplifying

$y_1 = 2$; $y_2 = -1 + i$; $y_3 = -1 - i$

Therefore the solutions of the general equation are:

$$x_1 = y_1 + 2 = 2 + 2 = 4$$

$x_2 = y_2 + 2 = -1 + i + 2 = 1 + i$

$x_3 = y_3 + 2 = -1 - i + 2 = 1 - i$

If $q^2 + p^3 = 0$, a and b are equal. So if m represents the common real value of a and b, we have

$y_1 = m+m = 2m$

$y_2 = -(m+m)/2 + (m-m)/2 (3\sqrt{i}) = -m$

$y_3 = -(m+m)/2 - (m-m)/2 (3\sqrt{i}) = -m$

Therefore, in this case we have that all roots are real and two are equal.

Example: Find the roots of $x^3 - 12x + 16 = 0$

We already have the reduced equation so,

$3p = -12 \quad p=-4$ and $p^3 = -64$

$2q = 16 \quad q=8$ and $q^2 = 64$ thus, $q^2 + p^3 = 64-64 = 0$. This means that the solution will have three real roots, two of them equal. The values of a and b are:

$$a= \sqrt[3]{-q + \sqrt{q^2 + p^3}} = \sqrt[3]{-8} = -2$$

$$b= \sqrt[3]{-q - \sqrt{q^2 + p^3}} = \sqrt[3]{-8} = - 2$$

Therefore the roots are:

$y_1 = a + b = -4$

$y_2 = a\partial + b\partial^2 = -2(\partial + \partial^2) = -2(-1/2 + (i\sqrt{3}/2) - 1/2 - (i\sqrt{3}/2) = 2$

$y_3 = a\partial^2 + b\partial = -2(\partial^2 + \partial) = -2(-1/2 - (i/3/2)\ \ - 1/2 - (i/3/2) = 2$

Now if $q^2 + p^3 < 0$, a and b will be complex numbers because of the square root of the negative discriminant. Therefore, if the values of a and b are a= M + Ni and b = M - Ni, where M = -q of the equation and N = $q^2 + p^3$, the solutions of the reduced equation are :

$$y_1 = a + b = 2M$$

$$y_2 = -2M/2 + 2Ni\ (\sqrt{3}i)/2 = -M - \sqrt{3}\ N$$

$$y_3 = -2M/2 - 2Ni\ (\sqrt{3}i)/2 = -M + \sqrt{3}\ N$$

which are all real roots and unequal.

However, there is no general arithmetic or algebraic method for finding the exact value of the cubic root of complex numbers. Therefore, Cardan's formula is of little use in this case, which is known as the irreducible. The irreducible case can be solved by using trigonometry but will not be addressed here.

For example: Solve the equation $y^3 -3y + 1 = 0$

It is already in the reduced cubic form so

$3p= -3$ $p = -1$ and $p^3 = -1$ and $2q = 1$, $q = 1/2$ and $q^2 = 1/4$

Therefore $p^3 + q^2 = -3/4$ which is less than 0. Trigonometry will have to used because of the negative discriminant.

Since we are now able to find the roots of a cubic equation, it makes it easier to graph. By finding the roots we can graph the x-intercepts of a cubic equation. Also, by finding the roots we can check our graphs when plotting by the conventional method.

As a teacher it is important to know when to introduce cubic equations into the classroom, if at all. The students should have a good background in the quadratic formula and their algebra skills should be strong. Of course, a first year algebra student would not be taught this topic. Finding the roots of a cubic equation by Cardan's formula can be a supplementary topic brought in at the end of a semester. An honors Algebra II class could be taught to use the formula one year at the end of the semester and the next year in Trigonometry they could find the solution for the irreducible case of the cubic equation where $q^2 + p^3 < 0$. Also, depending on the students knowledge of complex numbers, it may not be advisable to discuss complex numbers in the classroom.

Depending on whether or not the students have taken any computer classes and what languages they know, they could use a program to graph the cubic equation. If they have experience with the computer language BASIC they could type in the computer program supplied in the Appendix B. By typing out this program which graphs cubic equations, the students might be able to gain a little more insight into graphing cubics.

If Cardan's formula is not taught, a short proof can be done to show that a cubic equation can be solved by radicals. First, the teacher should show the students how to arrive at the the reduced cubic equation. The reduced cubic equation is in the form $y^3 + 3py + 2q = 0$. From here Cardan's formula can be given and illustrated. Hopefully, the students will notice that $q^2 + p^3$ is under a radical sign and is therefore the discriminant of the equation. The students' knowledge of complex numbers dictates whether or not they are taught that the roots of the equation are $y_1 = a + b$, $y_2 = a\partial + b\partial^2$, $y_3 = a\partial^2 + b\partial$ where $\partial = -1/2 + (\sqrt{3}/2)(i)$ and $\partial^2 = -1/2 - (\sqrt{3}/2)(i)$. Depending on the time that is available to teach the lesson the teacher can explain the cube roots of unity or merely tell them that ∂ is equal to the cube root of unity and simply substitute in the value for ∂ in the equation. It should be kept in mind that after finding 3p and 2q we then must find p,q,p^3 and q^2. After introducing Cardan's formula and the roots for y, a few examples could be given and then worked out on the board. There should be time for the students to do problems in class; however, homework will be assigned. If the students are familiar with Descartes' rule of signs it could be brought to their attention that they could check their answers using it. It should be realized that it will take a few days for the students to get a grasp on cubic equations. On tests and quizzes the students can find roots and graph the equations making sure to check x-intercepts and y-intercepts.

Conclusion

The topics in this paper can be presented in different ways according to the teacher's preference; however, due to the nature of the material the author thinks that the topics should be taught in the following order: synthetic division, remainder theorem ,Descartes' rule of signs and then cubic equations. Synthetic division and the remainder theorem could be taught in reverse order because they are related so closely but Descartes rule of signs and, especially cubic equations should be taught in the same order as this paper presents.

All the areas in the Theory of Equations can be used together to find solutions of roots to graph equations. Also, the theory of equations can help mathematicians at any level understand algebra a little better. In beginning calculus, functions and algebra play a major part, the topics discussed here will make students more comfortable when dealing with functions and equations. The topics listed here will help the reader have a deeper understanding of algebra and in turn can help ease entry into calculus.

The questions posed at the beginning of this paper are now answered. All of these topics can be presented in a high school classroom as supplementary material or as a means to help the students understand algebra better. Teachers and anyone else interested in mathematics should be familiar with The Theory of Equations because the subjects covered can be used in many different ways, from finding roots of an equation and transforming equations to giving a graphing procedure for real roots. After reading this paper the reader will have the same respect for the Theory of Equations as the author and can understand in part the significance of this topic in the world of mathematics.

APPENDIX A

Lesson Plan for Synthetic Division

Objective: Have the students learn to use synthetic division in the proper situation as an alternative to "long division".

Audience: The students should be high school students, who are capable of performing long division. The students should also be able to multiply, add and divide integers.

Instructional Plan: The teacher will introduce synthetic division by telling the students about synthetic division after the teacher is sure that the students can do long division. The teacher will put a particularly lengthy problem on the board and tell the students there is a faster and easier way to do this problem than long division. For example: $4x^9 - 5x^8 + 3x^7 - 2x^6 - 3x^5 + 9x^4 - 6x^3 - 2x^2 + 1$ dividied by $x-2$. The teacher then will go through the steps for synthetic division with the class on the board. The steps include 1. Arrange f(x) in descending powers of x; 2. Write coefficients a_i in order on a line. Supply a zero for each missing power of x ; 3. To divide by x-r write r at the left on the first line; 4. Show array and complete it

c]	a_0	a_1	a_2	a_3	a_{n-1}	a_n
		b_0r	b_1r	b_2r	$b_{n-2}r$	$b_{n-1}r$
	b_0	b_1	b_2	b_3	b_{n-1}	b_n

Where $b_0 = a_0$, $b_1 = a_1 + b_0r$, $b_2 = a_2 + b_1r, \ldots, b_i = a_i + b_{i-1}r$,

$$b_n = a_n + b_{n-1} = f(r) = R$$

The teacher then will use both methods, long division and synthetic division, to solve the problem given earlier in class. Next the students will do problems in class after another example with the teacher helping out with any problems. The teacher will then walk around the room while the students are doing problems and field any questions. Homework will be assigned and checked the next day. Throughout the course students will use synthetic division to solve various problems.

<u>Evaluation Method:</u> The teacher will gain knowledge to see how the students are doing with synthetic division by walking around. Also, hopefully the students can work problems on the board. The students will be tested on synthetic division when they are tested on the other operations on polynomials. Can students create their own problems when given the roots?

<u>Follow Up:</u> Throughout the semester the students will use synthetic division in various applications including finding rational roots.

Lesson Plan for Remainder Theorem

Objective: The students will learn how to use the Remainder theorem and Factor theorem to learn more about polynomial and their roots.

Audience: The students should be in high school and be proficient in factoring. The students should also know how to perform division on polynomials. This topic is one that should be addressed in an algebra class.

Instructional Plan: The student will be introduced to the topic by the teacher when he will tell them that they will be able to tell if a polynomial is divisible by (x-r). The teacher could do a short proof for example by showing that by definition $f(x) = (x-r)q(x) + R$. Next the teacher could parallel this with division with integers . It should be mentioned that the above polynomial Is In th form Dividend = divisor x quotient + remainder. The teacher then can ask the students what would happen if $R = 0$. By using the blackboard hopefully the teacher can direct the students to the conclusion that if $R = 0$ then (x-r) is a factor of f(x). Problems will be done on the board and students will be asked to participate in answering certain problems on the board. The teacher will explain that the Remainder theorem can be used as a means to check their work and help them find roots of equations very quickly. Homework will be given and hopefully students will have time to work in class on problems so the teacher can answer any questions.

Evaluation Method: Did students participate in answering questions on the blackboard? How did they do on the homework? When tested did the students do well?

Follow Up: Throughout the course students will be referred back to the remainder and factor theorem. In the next few classes the teacher should ask questions to make sure the students still understand the material.

Lesson Plan for Descartes' Rule of Signs

Objective: The Students will learn Descartes' rule of signs and will apply it to solving equations.

Audience: High School Students who have taken algebra and have dealt with equations of the form

$$a_0x^n + a_1x^{n-1} + ... + a_n = 0$$

The students should know how to find roots of at least quadratic equations. The topic can be taught to first year algebra students.

Instructional Plan: Give the students a sequence of integers like -2,-3,4,4,-1,7,7,7,-5,-4,1. Tell them to count how many times the sign changes then how many times it stays the same. The teacher should then put the numbers in front of variables to create an equation. The teacher should then explain that every time two consecutive terms have the same sign it is called a permanence and when the signs change it is called a variation. The teacher can then state Descartes' rule of signs and explain to the student that if any of the coefficients are 0 then the next terms in front of or behind the zero are considered consecutive terms. An example of this equation can be $4x^5 + 4x^4 - 2x + 1$. The coefficients can listed 4,4,0,0,-2,1 and the students will be asked to tell how many variation there are. After the teacher is sure that the students understand the variations and permanences the teacher can then explain that the number of real roots of an equation with real coefficients is equal to or less than the number of variations. Next, the teacher can then explain to the students that for f(-x) the number of variations is equal to or less than by an even number of the negative roots. A few examples will be given starting with the most basic. Example: $x-4=0$, we know $V=1$ for f(x) so one positive root and for f(-x) there is not a variation sot there is only one postive roots which we know is $x=4$. From this example more will be given and problems will be done in class. Since the topic is fairly easy to understand hopefully applications can be given during class and homework will be assigned. The rational roots theorem will be introduced along with the thought that imaginary roots occur in pairs to give the student a better idea of the topic.

Evaluation Method: Did the students respond in class? Could they answer questions on the material? Were the homework assignments done correctly? These questions along with a subsequent quiz can be given to test the students' knowledge on the material.

Follow Up: In the next few lessons and throughout the semester students will be using Descartes' rule to find roots of equations. Teacher can use Descartes' rule when using Cardan's Formula for cubic equations.

Lesson Plan for Cubic Equations

Objective: The students will learn how to use Cardans' Formula to solve cubic equations.

Audience: The students should be very proficient in mathematics and skilled in using the quadratic formula. The students should have had experience with complex numbers and should be strong in algebra.

Instructional Plan: The students will be introduced to the material by the teacher showing them examples of the quadratic equation and asking them if they think it is possible to just use coefficients to find the roots of a cubic equation. The teacher will then give the students Cardan's Formula $y=\sqrt[3]{-q + q^2 + p^3} + \sqrt[3]{-q - q^2 + p^3}$. Next, the teacher will explain that to use this formula the students have to put the cubic equation in reduced cubic equation form which is $y^3 + 3py + 2q = 0$. To arrive at this equation it should be stated that the original cubic equation $ax^3 + bx^2 + cx + d = 0$ should be changed by making a substitution $x = y - b/3a$. Once the values for p and q are found by making the substitution, the roots can be given as $y_1 = a + b$, $y_2 = a\partial + b\partial^2$ and $y_3 = a\partial^2 + b\partial$ where ∂ is a cube root of unity where the value for $\partial = -1/2 + 3/2(i)$ and $\partial 2 = -1/2 - 3/2(i)$. The students should separate a and b and then substitute into the equation. A few examples should be done on the board and the teacher should take questions. If there is enough time in class the students can work on problems. The teacher should tell the students to notice that $q^2 + p^3$ is the discriminant of the equation because it is underneath the square root sign. Depending on the amount of time available in class a study of the discriminant could be made in this class period or another one.

Evaluation Method: How did the students do on the problems assigned in class? Did they have any problems on the homework assignments? A quiz or test on the material will be given.

Follow Up: Now that students can find the roots of an equation they should be able to graph cubic equations and find the x and y intercepts. In other class periods a study of the discriminant of the equation can be done.

Computer Program to Graph Cubic Polynomials

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1     REM*** A COMPUTER PROGRAM TO GRAPH CUBIC POLYNOMIALS BY
      HARVEY BRAVERMAN. (C) 1986.  PERMISSION IS GRANTED TO COPY
      THIS PROGRAM.
150   TEXT: HOME
160   PRINT: PRINT: PRINT: "PLEASE CHOOSE THE RANGE OF INTEGERS":
      PRINT : PRINT: "THAT YOU WANT FOR THE ROOTS OF THE":
      PRINT: PRINT "CUBIC POLYNOMIAL."
170   VTAB 8 : PRINT : PRINT : PRINT"FIRST ROOT FROM " :  GOSUB 1130:
      VTAB: 10 : HTAB: INPUT A1: GOSUB 1140
180   IF INT(A1) <> A1 THEN VTAB 10: HTAB 1: PRINT SPC(40): GOTO 170
190   GOSUB 1130: VTAB 10: HTAB 25: PRINT "TO": VTAB 10: HTAB 30: INPUT
      A2: GOSUB 1140
200   IF INT (A2) <> A2 TH VTAB 10: HTAB 30: PRINT SPC (10): GOTO 190
210   IF A2< A1  THEN VTAB 12: PRINT "CHOOSE A LARGER NUMBER.": VTAB
      10: HTAB 30: PRINT SPC (10) : GOTO 190
220   HOME
230   VTAB 8: PRINT: PRINT : PRINT "SECOND ROOT FROM" : GOSUB 1130:
      VTAB 10: HTAB 18: INPUT B1: GOSUB 1140
240   IF INT(B1) <> B1 THEN VTAB 10: HTAB 1: PRINT SPC(40): GOTO 230
250   GOSUB 1130: VTAB 10: HTAB 25: PRINT "TO": VTAB 10: HTAB 30: INPUT
      B2: GOSUB 1140
260   IF INT (B2) <>B2 THEN VTAB 10: HTAB 30: PRINT SPC(10): GOTO 250
270   IF B2<B1 THEN VTAB 12: PRINT "CHOOSE A LARGER NUMBER.": VTAB
      10: HTAB 30: PRINT SPC (10): GOTO 250
280   HOME
290   VTAB 8: PRINT : PRINT: PRINT "THIRD ROOT FROM ": GOSUB 1130:VTAB
      10: HTAB 18: INPUT C1: GOSUB 1140
300   IF INT (C1) < > C1 THEN VTAB 10: HTAB 1: PRINT SPC(40): GOTO 290
310   GOSUB 1130: VTAB 10 : HTAB 25: ORINT "TO" ; VTAB 10: HTAB 30:
      INPUT C2: GOSUB 1140
320   IF INT (C2) <> C2 THEN VTAB 10: HTAB 30: PRINT SPC(10): GOTO 310
330   IF C2<C1 THEN VTAB 12: PRINT "CHOOSE A LARGER NUMBER.": VTAB
      10: HTAB 30: PRINT SPC( 10): GOTO 310
340   HOME
520   FOR A = A1 TO A2
540   FOR B = B1 TO B2
```

```
560  FOR C = C1 TO C2
580  S = ( A + B + C) /3
600  R = (SQR (A * (A-B) + B * (B-C) + C * (C-A))) / 3
620  IF ABS (S - INT (S)) . .1 GOTO 1000
640  IF ( R - INT ( R)) > .001 GOTO 1000
660  IF R = 0 GOTO 1000
690  E = FRE (0)
700  IF ( A+ B+ C) > 0 THEN S$ = "-" + STR$ ((A + B+C)) + "X"
720  IF (A +B + C) < 0 THEN S$= "+" STR$ ( ABS( A + B + C)) + "X"
740  IF ( A + B + C) = 0 THEN S$= ""
760  IF ((A * B) + (B*C) + (A *C)) > 0 THEN T$ = " + " + STR$ (((A*B) + (B*C)+
     (A*C))) + "X"
780  IF ((A * B)  + (B * C) + ( A * C)) <0 THEN T$ = "-" + STR$ ( ABS ((A*B) +
     (B*C) + ( A *C))) + "X"
800  IF (( A * B) + (B * C) + ( A * C)) = 0 THEN T$= ""
820  IF A*B *C> 0 THEN F$ = "- " + STR$ ( A* B * C)
830  IF A * B * C , 0 THEN F$ = " + " + STR$ ( ABS (A * B* C))
840  IF A * B* C = 0 THEN F$ = ""
850  PR#1: PRINT "***        ROOTS: ";A; "," ;B; ", ";C:
852  IF S$ < > "" THEN PRINT "3"; SPC(    LEN ( S$)); "2"
854  IF S$= "" THEN PRINT " 3"
860  PRINT "X" ; S$; T$; F$: PRINT
865  X = S- R
870  IF ABS ( X) < .1 THEN X = 0
875  IF LEN ( STR$ ( X)) > 6 THEN X = SGN ( X) * INT (( ABS ( X) + .001) *
     100) /100
880  PRINT "MAX(";X;",";
890  GOSUB 110
893  X = S + R
895  IF ABS (X) < .1 THEN X = 0
897  IF LEN ( STR$ ( X)) > 6 THEN X = SGN (X) * INT ((ABS(X) + .001)
     *100/100
900   PRINT Y; "0 MIN(";X;",";
910  GOSUB 1100
920  PRINT Y; ") PT INF ( "; S; ",";
930  X = S: GOSUB 1100
940   PRINT Y; "0"; PRINT
950  PR# 0 : PRINT
1000 NEXT C
1020 NEXT B
```

```
1040 NEXT A
1050  PRINT : PRINT : PRINT " DONE"; END
1100 Y = X^3 - (( A + B + C) * (X^2)) + (((A*B) + ( A *C) + (B * C)) * X) -
     (A * B * C)
1110  IF ABS (Y) < .1 THEN Y = 0
1115 IF LEN ( STR$ ( Y)) > 6 THEN Y = SGN ( Y) * INT (( ABS (Y) + .001) *100
     /100
1120 RETURN
1130 VTAB 14: HTAB 1: PRINT " PRESS <RETURN> TO CONTINUE.": RETURN
1140  VTAB 14:  HTAB 1: PRINT SPC(40)
1150 RETURN
```

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